

# On elliptic solutions of the cubic complex one-dimensional Ginzburg–Landau equation

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## Abstract

The cubic complex one-dimensional Ginzburg–Landau equation is considered. Using the Hone's method, based on the use the Laurent-series solutions and the residue theorem, we have proved that this equation has neither elliptic standing wave nor elliptic travelling wave solutions. This result amplifies Hone's result, that this equation has no elliptic travelling wave solutions.

*Key words:* Standing wave, elliptic function, residue theorem, the cubic complex one-dimensional Ginzburg–Landau equation

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## 1 Introduction

The one-dimensional cubic complex Ginzburg–Landau equation (CGLE) [11] is one of the most-studied nonlinear equations (see [4] and references therein). It is a generic equation which describes many physical phenomena, such as pattern formation near a supercritical Hopf bifurcation [4,10], the propagation of a signal in an optical fiber [2], spatiotemporal intermittency in spatially extended dissipative systems [15,20].

The CGLE

$$iA_t + pA_{xx} + q|A|^2A - i\gamma A = 0, \quad (1)$$

where subscribes denote partial derivatives:  $A_t \equiv \frac{\partial A}{\partial t}$ ,  $A_{xx} \equiv \frac{\partial^2 A}{\partial x^2}$ ,  $p \in \mathbb{C}$ ,  $q \in \mathbb{C}$  and  $\gamma \in \mathbb{R}$  is not integrable if  $pq\gamma \neq 0$ . In the case  $q/p \in \mathbb{R}$ ,  $\gamma = 0$  the

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CGLE is integrable and coincides with the well-known nonlinear Schrödinger equation [13,22,25].

One of the most important direction in the study of the CGLE is the consideration of its travelling wave reduction [3,4,6,8,16,18,19,21,26]:

$$A(x, t) = \sqrt{M(\xi)} e^{i(\varphi(\xi) - \omega t)}, \quad \xi = x - ct, \quad c \in \mathbb{R}, \quad \omega \in \mathbb{R}, \quad (2)$$

which defines the following third order system

$$\begin{cases} \frac{M''}{2M} - \frac{M'^2}{4M^2} - \left(\psi - \frac{cs_r}{2}\right)^2 - \frac{cs_i M'}{2M} + d_r M + g_i = 0, \\ \psi' + \left(\psi - \frac{cs_r}{2}\right) \left(\frac{M'}{M} - cs_i\right) + d_i M - g_r = 0, \end{cases} \quad (3)$$

where  $\psi \equiv \varphi' \equiv \frac{d\varphi}{d\xi}$ ,  $M' \equiv \frac{dM}{d\xi}$ , six real parameters  $d_r$ ,  $d_i$ ,  $g_r$ ,  $g_i$ ,  $s_r$  and  $s_i$  are given in terms of  $c$ ,  $p$ ,  $q$ ,  $\gamma$  and  $\omega$  as

$$d_r + id_i = \frac{q}{p}, \quad s_r - is_i = \frac{1}{p}, \quad g_r + ig_i = \frac{\gamma + i\omega}{p} + \frac{1}{2}c^2 s_i s_r + \frac{i}{4}c^2 s_r^2. \quad (4)$$

Using (3) one can express  $\psi$  in terms of  $M$  and its derivatives:

$$\psi = \frac{cs_r}{2} + \frac{G' - 2cs_i G}{2M^2(g_r - d_i M)}, \quad (5)$$

where

$$G \equiv \frac{1}{2}MM'' - \frac{1}{4}M'^2 - \frac{cs_i}{2}MM' + d_r M^3 + g_i M^2, \quad (6)$$

and obtain the third order equation in  $M$ :

$$(G' - 2cs_i G)^2 - 4GM^2(d_i M - g_r)^2 = 0. \quad (7)$$

We will consider the case

$$\frac{p}{q} \notin \mathbb{R}. \quad (8)$$

In this case equation (7) is not integrable [7,8], which means that the general solution (which should depend on three arbitrary integration constants) is not known. Using the Painlevé analysis [7] or topological arguments [26] it

has been shown that single-valued solutions can depend on only one arbitrary parameter. Equation (7) is autonomous, so this parameter is  $\xi_0$ : if  $M = f(\xi)$  is a solution, then  $M = f(\xi - \xi_0)$ , where  $\xi_0 \in \mathbb{C}$  has to be a solution. Special solutions in terms of elementary functions have been found in [5,8,19,23]. All known exact solutions of the CGLE are elementary (rational, trigonometric or hyperbolic) functions. The full list of these solutions is presented in [18,21].

In [21] a new method to search single-valued particular solutions has been developed. Rather than looking for an explicit, closed form expression, R. Conte and M. Musette look for the first order polynomial autonomous ODE for  $M(\xi)$ . This method allows to find either elliptic or elementary solutions. It is based on the Painlevé analysis [24] and uses the formal Laurent-series solutions. Using these solutions A.N.W. Hone [18] has proved that a necessary condition for eq. (7) to admit elliptic solutions is  $c = 0$ . The goal of this paper is to prove that eq. (7) does not admit elliptic solutions in the case  $c = 0$  as well. In other words, neither travelling nor standing wave solutions are elliptic functions. In contrast to [21,18] we consider system (3) instead of eq. (7). Below we show that this choice has some preferences. We consider not only generic (non-zero) values of parameters but also these zero values. The condition (8) gives only one restriction:  $d_i \neq 0$ .

## 2 Elliptic functions

The function  $\varrho(z)$  of the complex variable  $z$  is a doubly-periodic function if there exist two numbers  $\omega_1$  and  $\omega_2$  with  $\omega_1/\omega_2 \notin \mathbb{R}$ , such that for all  $z \in \mathbb{C}$

$$\varrho(z) = \varrho(z + \omega_1) = \varrho(z + \omega_2). \quad (9)$$

By definition a double-periodic meromorphic function is called an elliptic function. These periods define the period parallelograms with vertices  $z_0$ ,  $z_0 + N_1\omega_1$ ,  $z_0 + N_2\omega_2$  and  $z_0 + N_1\omega_1 + N_2\omega_2$ , where  $N_1$  and  $N_2$  are arbitrary natural numbers and  $z_0$  is an arbitrary complex number. The classical theorems for elliptic functions (see, for example [12,17]) prove that

- If an elliptic function has no poles then it is a constant.
- The number of elliptic function poles within any finite period parallelogram is finite.
- The sum of residues within any finite period parallelogram is equal to zero (**the residue theorem**).
- If  $\varrho(z)$  is an elliptic function then any rational function of  $\varrho(z)$  and its derivatives is an elliptic function as well.

From (5) it follows that if  $M$  is an elliptic function then  $\psi$  has to be an elliptic function. Therefore, if we prove that  $\psi$  can not be an elliptic function, we prove that  $M$  can not be an elliptic function as well. To prove this we construct the Laurent-series solutions for system (3) and apply the residue theorem to the function  $\psi$  and its degrees.

### 3 Nonexistence of the standing wave elliptic solutions

#### 3.1 The Laurent-series solutions and the residue theorem

To prove the non-existence of elliptic solutions to (3) we will use its solutions in the form of the Laurent series, which can be easily found due to the Ablowitz–Ramani–Segur algorithm of the Painleve test [1]. In such a way we obtain solutions only as a formal series, but really we will use only a finite number of these series coefficients, so, we do not need the convergence of these series. It is known [8,18,21] that there are only two types of the Laurent-series solutions of (3) or (7). These solutions depend on only one arbitrary parameter  $\xi_0$ , which determines the position of the singular point. At singular points  $\psi$  and  $M$  tend to infinity as  $1/t$  and  $1/t^2$  respectively. We will use the Laurent series for the function  $\psi$ :

$$\psi_1 = \sum_{k=-1}^{\infty} C_k(\xi - \xi_0)^k \quad \text{and} \quad \psi_2 = \sum_{k=-1}^{\infty} D_k(\xi - \tilde{\xi}_0)^k, \quad (10)$$

with  $C_{-1} \neq 0$  and  $D_{-1} \neq 0$ . A nonconstant elliptic function should have poles. Let  $\psi(\xi)$  in some parallelogram of periods has  $N_1 + N_2$  poles, its Laurent series expansions are  $\psi_1$  in the neighbourhood of  $N_1$  poles and are  $\psi_2$  in the neighbourhood of  $N_2$  poles. If  $\psi(\xi)$  is an elliptic function then the sum of its residues in some parallelogram of periods has to be zero, therefore, this function has both types of the Laurent series expansions (10) and

$$N_1 = -\frac{D_{-1}}{C_{-1}}N_2. \quad (11)$$

If  $\psi(\xi)$  is an elliptic function then powers  $\psi^k$  have to be elliptic functions as well, they have  $N_1$  Laurent series expansions  $\psi_1^k$  and  $N_2$  Laurent series expansions  $\psi_2^k$ . To calculate residues of  $\psi_1^k$  (or  $\psi_2^k$ ) we have to use only  $k$  leading terms of the Laurent series  $\psi_1$  ( $\psi_2$ ). The residue theorem for the functions  $\psi(\xi)^k$  gives algebraic equations on the coefficients of  $\psi_1$  and  $\psi_2$  Laurent series. These series depend on numerical parameters of system (3) and only on them (have no resonances), hence, we obtain a system of algebraic equations on

coefficients of (3), at which (3) can have elliptic solutions. For example if we demand that the functions  $\psi^2$ ,  $\psi^3$ ,  $\psi^4$  and  $\psi^5$  are elliptic, then, using (11), we obtain the following system on  $C_k$  and  $D_k$ :

$$\begin{cases} C_0 = D_0, \\ C_1 C_{-1} + C_0^2 = D_1 D_{-1} + D_0^2, \\ C_2 C_{-1}^2 + 3C_1 C_0 C_{-1} + C_0^3 = D_2 D_{-1}^2 + 3D_1 D_0 D_{-1} + D_0^3, \\ C_3 C_{-1}^3 + 4C_2 C_0 C_{-1}^2 + 2C_1^2 C_{-1}^2 + 6C_{-1} C_0^2 C_1 + C_0^4 = \\ = D_3 D_{-1}^3 + 4D_2 D_0 D_{-1}^2 + 2D_1^2 D_{-1}^2 + 6D_1 D_0^2 D_{-1} + D_0^4. \end{cases} \quad (12)$$

We will use also the corresponding equation for  $\psi^7$  under conditions  $C_0 = 0$ ,  $C_2 = 0$ ,  $C_4 = 0$ ,  $D_0 = 0$ ,  $D_2 = 0$  and  $D_4 = 0$ :

$$C_5 C_{-1}^5 + 6C_3 C_1 C_{-1}^4 + 5C_1^3 C_{-1}^3 = D_5 D_{-1}^5 + 6D_{-1}^4 D_3 D_1 + 5D_1^3 D_{-1}^3. \quad (13)$$

We have calculated the residues of powers of  $\psi$  with the help of the procedure **ydegree** from our package of Maple [14] procedures [27], which realizes the Conte–Musette algorithm for construction of single-valued solutions of nonintegrable systems [21].

### 3.2 The number of essential numerical parameters of system (3)

System (3) includes seven arbitrary constants, some of them can be fixed without loss of generality. First of all one can fix  $s_r$  and  $s_i$ . From the condition  $p \notin \mathbb{R}$  (the case of real  $p$  we consider separately) follows that  $s_i \neq 0$ . Using the following transformations:

$$\tilde{c} = \varpi c, \quad \tilde{s}_i = \frac{s_i}{\varpi}, \quad \tilde{s}_r = \tau s_r, \quad \tilde{\psi} = \psi - \frac{cs_r}{2}(1 - \tau\varpi) \quad (14)$$

one can put

$$\tilde{s}_r = -\frac{1}{10} \quad \text{and} \quad \tilde{s}_i = -\frac{3}{10}. \quad (15)$$

Using the transformations

$$\tilde{M} = \mu M, \quad \tilde{d}_i = \frac{d_i}{\mu}, \quad \tilde{d}_r = \frac{d_r}{\mu} \quad (16)$$

we can fix  $d_r$  or  $d_i$ . Following [18] we will fix the value of  $d_r$ . Our restriction (8) on parameters  $p$  and  $q$  gives no information about  $d_r$ , so we have to consider two cases:  $d_r = 0$  and  $d_r \neq 0$  separately. Using scaling transformations of the independent variable  $\xi$  it is possible to fix  $g_i$  or  $g_r$ , but, following [21,18] we leave them arbitrary to consider zero and nonzero values of these parameters at once.

From the second equation of (3) it follows that if  $\psi$  is a constant then  $M$  can not be an elliptic function, so to obtain nontrivial elliptic solutions we have to assume that  $\psi$  has poles. We do not restrict ourself to the case  $c = 0$  and prove the non-existence of either travelling or standing wave solutions. It has been noted in [9] that one does not need to transform a system of differential equations into one equation to obtain the Laurent-series solutions.

### 3.3 The case $d_r = 0$

Let us consider system (3) with

$$d_r = 0, \quad s_r = -\frac{1}{10} \quad \text{and} \quad s_i = -\frac{3}{10}. \quad (17)$$

From the condition (8) it follows that  $d_i \neq 0$ , therefore, there exist two different Laurent-series solutions ( $\xi_0 = 0$ ) of (3):

$$\check{\psi}_1 = \frac{\sqrt{2}}{\xi} - \frac{c(\sqrt{2}+1)}{20} + \mathcal{O}(\xi), \quad (18)$$

$$\check{M}_1 = \frac{3\sqrt{2}}{d_i} \left( \frac{1}{\xi^2} - \frac{1}{10\xi} \right) + \mathcal{O}(1) \quad (19)$$

and

$$\check{\psi}_2 = -\frac{\sqrt{2}}{\xi} + \frac{c(\sqrt{2}-1)}{20} + \mathcal{O}(\xi), \quad (20)$$

$$\check{M}_2 = -\frac{3\sqrt{2}}{d_i} \left( \frac{1}{\xi^2} - \frac{1}{10\xi} \right) + \mathcal{O}(1) \quad (21)$$

From (11) it follows that  $N_1 = N_2$ , that is to say, if the function  $\psi(\xi)$  has  $N$  poles with residues, which are equal to  $\sqrt{2}$ , within some finite period parallelogram, then in this domain the number of poles, which residues are equal to  $-\sqrt{2}$ , has to be equal to  $N$  as well.

Residues of  $\check{\psi}_1^2$  are equal to  $-2\sqrt{2}c(\sqrt{2}+1)/20$ , whereas residues of  $\check{\psi}_2^2$  are  $-2\sqrt{2}c(\sqrt{2}-1)/20$ . From the first equation of system (12) we obtain that the sum of residues of the function  $\psi^2$  is equal to zero if and only if  $c = 0$ . So, we prove the absence of the travelling wave solutions. Note that to obtain this result we have used only two coefficients of the Laurent series  $\psi_1$  and  $\psi_2$ . In the case  $c = 0$  we have to apply the residue theorem for  $\psi_3$  and  $\psi_4$ , so, we have to calculate four coefficients in these series (two of them are zero at  $c = 0$ )

$$\check{\psi}_1 = \frac{\sqrt{2}}{\xi} + \frac{0}{\xi} + \frac{1}{21} (5\sqrt{2}g_i - g_r) \xi + 0\xi^2 + \mathcal{O}(\xi^3), \quad (22)$$

and

$$\check{\psi}_2 = -\frac{\sqrt{2}}{\xi} + \frac{0}{\xi} - \frac{1}{21} (5\sqrt{2}g_i + g_r) \xi + 0\xi^2 + \mathcal{O}(\xi^3), \quad (23)$$

From the second and the third equations of (12) we obtain that the functions  $\psi^3$  and  $\psi^4$  satisfy the residue theorem if and only if

$$g_i = 0 \quad \text{and} \quad g_r = 0. \quad (24)$$

In this case the Laurent-series solutions give

$$\check{\psi}_1(\xi) = \frac{\sqrt{2}}{\xi}, \quad \check{M}_1(\xi) = \frac{3\sqrt{2}}{d_i\xi^2} \quad (25)$$

and

$$\check{\psi}_2(\xi) = -\frac{\sqrt{2}}{\xi}, \quad \check{M}_2(\xi) = -\frac{3\sqrt{2}}{d_i\xi^2}. \quad (26)$$

The straightforward substitution of these functions in system (3) with  $c = 0$ ,  $d_r = 0$ ,  $g_r = 0$  and  $g_i = 0$  proves that they are exact solutions. Using the Ablowitz–Ramani–Segur algorithm [1] it is easy to prove that the coefficients of the Laurent-series solutions does not include arbitrary parameters, so the obtained solutions are unique single-valued solutions and the CGLE has no elliptic solution for these values of parameters as well. Thus we have proved the non-existence of both travelling and standing wave elliptic solutions at  $d_r = 0$ .

### 3.4 The case $d_r \neq 0$

In this case we can use the following values of parameters without loss of generality

$$d_r = \frac{1}{2}, \quad s_r = -\frac{1}{10} \quad \text{and} \quad s_i = -\frac{3}{10}. \quad (27)$$

To simplify calculations we, following [18], express  $d_i$  through a new real parameter:

$$d_i = \pm \frac{3\sqrt{\beta^2 - 1}}{4\sqrt{2}}, \quad \beta > 1. \quad (28)$$

If  $d_i > 0$  (the sign  $+$  in (28)) then system (3) has the following Laurent series solutions:

$$\tilde{\psi}_1 = \frac{\sqrt{2}(\beta + 1)}{\sqrt{\beta^2 - 1}}\xi^{-1} - \frac{c}{10} \left( \frac{\sqrt{\beta^2 - 1}}{\sqrt{2}(\beta - 1)} + \frac{1}{2} \right) + \mathcal{O}(\xi) \quad (29)$$

$$\tilde{\psi}_2 = -\frac{\sqrt{2}(\beta - 1)}{\sqrt{\beta^2 - 1}}\xi^{-1} + \frac{c}{10} \left( \frac{\sqrt{\beta^2 - 1}}{\sqrt{2}(\beta + 1)} - \frac{1}{2} \right) + \mathcal{O}(\xi) \quad (30)$$

If there are  $N_1$  Laurent series of type  $\tilde{\psi}_1$  and  $N_2$  Laurent series of type  $\tilde{\psi}_2$  then eq. (11) gives

$$N_1 = \frac{\beta - 1}{\beta + 1} N_2. \quad (31)$$

The residues theorem for  $\psi^2$  gives  $c\beta = 0$ . Using condition  $\beta > 1$ , we derive that  $c$  has to be equal to zero, so we reobtain the main result of [18] that the CGLE has no elliptic travelling wave solutions for non-zero values of parameters. Note that the use of Laurent series of  $\psi(\xi)$  instead of the Laurent series of  $M(\xi)$  allows to simplify calculations.



Let us consider the standing wave solutions ( $c = 0$ ) of the CGLE. The Laurent series solutions are:

$$\begin{aligned}\tilde{\psi}_1 = & \frac{\sqrt{2}(\beta+1)}{\sqrt{\beta^2-1}}\xi^{-1} - \frac{(\beta g_r - 5g_r - 5\sqrt{2(\beta^2-1)}g_i)}{3(7\beta+5)}\xi + \\ & + \frac{1}{90(\beta+1)(7\beta+5)^2} \left\{ 32\beta^3 g_i g_r - 256\beta^2 g_i g_r - 32\beta g_i g_r + \right. \\ & + 256g_i g_r + \sqrt{2(\beta^2-1)} (122\beta^2 g_i^2 + 11\beta^2 g_r^2 - 34\beta g_r^2 + \\ & \left. + 61g_r^2 - 122g_i^2) \right\} \xi^3 + \mathcal{O}(\xi^5),\end{aligned}\tag{32}$$

$$\begin{aligned}\tilde{\psi}_2 = & -\frac{\sqrt{2}(\beta-1)}{\sqrt{\beta^2-1}}\xi^{-1} - \frac{(\beta g_r + 5g_r + 5\sqrt{2(\beta^2-1)}g_i)}{3(7\beta-5)}\xi + \\ & + \frac{1}{90(\beta+1)(7\beta+5)^2} \left\{ 32\beta^3 g_i g_r + 256\beta^2 g_i g_r - 32\beta g_i g_r - \right. \\ & - 256g_i g_r + \sqrt{2(\beta^2-1)} (122\beta^2 g_i^2 - 11\beta^2 g_r^2 + 34\beta g_r^2 + \\ & \left. + 61g_r^2 - 122g_i^2) \right\} \xi^3 + \mathcal{O}(\xi^5).\end{aligned}\tag{33}$$

The residues of  $\psi^2$ ,  $\psi^4$  and  $\psi^6$  are equal to zero at  $c = 0$ . Substituting the coefficients of the Laurent series  $\tilde{\psi}_1$  and  $\tilde{\psi}_2$ , we transform system (12) and eq. (13) into the algebraic system in  $\beta$ ,  $g_i$  and  $g_r$ . This system is too cumbersome to be presented here. The condition  $\beta > 1$  leaves only one solution of this system:

$$g_r = 0, \quad g_i = 0.\tag{34}$$

In the case  $d_i < 0$  we also obtain that the residue theorem for powers of  $\psi$  can be satisfied only if  $g_r = 0$  and  $g_i = 0$ . Let us consider system (3) with zero values of  $c$ ,  $g_i$  and  $g_r$ ,  $d_r = 1/2$  and an arbitrary (nonzero) value of  $d_i$ :

$$\begin{cases} 2MM'' - M'^2 - 4M^2\psi^2 + 2M^3 = 0, \\ M\psi' + M'\psi + d_i M^2 = 0, \end{cases}\tag{35}$$

The straightforward substitution gives that functions

$$\tilde{\psi}_1(\xi) = \frac{3 + \sqrt{9 + 32d_i^2}}{4d_i\xi}, \quad \tilde{M}_1(\xi) = \frac{3(3 + \sqrt{9 + 32d_i^2})}{4d_i^2\xi^2}\tag{36}$$

and

$$\tilde{\psi}_2(\xi) = \frac{3 - \sqrt{9 + 32d_i^2}}{4d_i\xi}, \quad \tilde{M}_2(\xi) = \frac{3(3 - \sqrt{9 + 32d_i^2})}{4d_i^2\xi^2} \quad (37)$$

are exact solutions of system (35). This system has no other single-valued solutions, so we have proved the non-existence of neither elliptic standing wave nor elliptic travelling wave solutions in case  $d_r \neq 0$  as well. In our calculations we assume that  $s_i \neq 0$ . At the same time our results prove the non-existence of elliptic solutions in the case  $s_i = 0$  too. Indeed, if  $c = 0$  then cases with  $s_i = 0$  and  $s_i \neq 0$  coincide, to transform the case  $\{s_i = 0, c \neq 0\}$  into the considered case  $\{s_i \neq 0, c = 0\}$  we have to add a constant to  $\psi(\xi)$ .

## 4 Conclusions

The Laurent-series solutions are useful not only to find elliptic solutions, but also to prove the non-existence of them. Using the Hone's method, based on residue theorem, we have proved the non-existence of both standing and travelling wave elliptic solutions of the CGLE in the case  $p/q \notin \mathbb{R}$ . Our result amplifies the Hone's result [18], that the CGLE with generic (non-zero) values of parameters has no elliptic travelling wave solution.

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